UNIT-I
THE RANDOM VARIABLE

The Random Variable: Concept of random variable, Condition for a function to be a random variable, Classification of a random variable, Cumulative distribution function and properties, Probability density function and properties, Different distributions-Binomial, Poisson, Uniform, Exponential, Rayleigh, Gaussian.

Operations on One Random Variable: Expectation, Moments about the origin, Central moments, Variance, Skew, Skewness, Characteristic function and properties, Moment generating function and properties, Transformations of a random variable.

PROBABILITY

- ✓ Probability theory is used in all those situations where there is randomness about the occurrence of an event.

  (Or)

  Measure of the likeliness that an event will occur.
- ✓ The meaning of randomness is existence of certain amount of unsureness about an event.
- ✓ This section deals with the chances of occurring any particular phenomena. i.e. Electron emission, Telephone calls, Radar detection, quality control, system failure, games of chance, birth and death rates, Random walks, probability of detection, Probability of false alarm, BER calculation, optimal coding (Huffman) and many more.

  Experiment:-

  A *random experiment* is an action or process that leads to one of several possible outcomes

Sample Space:

The sample space is the collection of all possible outcomes of a random experiment. The elements of are called **sample points**.

- A sample space may be finite, countably infinite or uncountable.
- A finite or countably infinite sample space is called a discrete sample space.
- An uncountable sample space is called a continuous sample space

Types of Sample Space:

Finite/Discrete Sample Space:

Consider the experiment of tossing a coin twice. The sample space can be S = {HH, HT, T H , TT} the above sample space has a finite number of sample points. It is called a finite sample space.

Countably Infinite Sample Space:

- ✓ Consider that a light bulb is manufactured. It is then tested for its life length by inserting it into a socket and the time elapsed (in hours) until it burns out is recorded.
- ✓ Let the measuring instrument is capable of recording time to two decimal places, for example 8.32 hours.
- ✓ Now, the sample space becomes count ably infinite i.e.
  S = {0.0, 0.01, 0.02}
  The above sample space is called a countable infinite sample space.

Uncountable / Infinite Sample Space/Continuous Sample Space

If the sample space consists of unaccountably infinite number of elements then it is called uncountable/ Infinite Sample Space.

Event

An event is simply a set of possible outcomes. To be more specific, an event is a subset A of the sample space S.

Example 1: tossing a fair coin

Example 2: Throwing a fair die

Types of Events:

Exhaustive Events:

- ✓ A set of events is said to be exhaustive, if it includes all the possible events.

  Ex. In tossing a coin, the outcome can be either Head or Tail and there is no other possible outcome. So, the set of events { H , T } is exhaustive.

Mutually Exclusive Events:

- ✓ Two events, A and B are said to be mutually exclusive if they cannot occur together.

  If two events are mutually exclusive then the probability of either occurring is

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

  Ex. In tossing a die, both head and tail cannot happen at the same time.

Equally Likely Events:

- ✓ Each outcome of the random experiment has an equal chance of occurring.

Ex. In tossing a coin, the coming of the head or the tail is equally likely.

Independent Events:

- ✓ Two events are said to be independent, if happening or failure of one does not affect the happening or failure of the other. Otherwise, the events are said to be dependent.
- ✓ If two events, *A* and *B* are independent then the joint probability is

$$P(A \text{ and } B) = P(A \cap B) = P(A)P(B),$$

Axioms of Probability

For any event *A*, we assign a number *P*(*A*), called the probability of the event *A*. This number satisfies the following conditions that act the axioms of probability.

(i) $P(A) \geq 0$ (Probabili ty is a nonnegativ e number)

(ii) $P(s) = 1$ (Probabili ty of the whole set is unity)

(iii) If $A \cap B = \phi$, **then** $P(A \cup B) = P(A) + P(B)$.

Note that (iii) states that if *A* and *B* are mutually exclusive (M.E.) events, the probability of their union is the sum of their probabilities

Relative Frequency

- ✓ Random experiment with sample space S. we shall assign non-negative number called probability to each event in the sample space.
- ✓ Let A be a particular event in S. then “the probability of event A” is denoted by P(A).
- ✓ Suppose that the random experiment is repeated n times, if the event A occurs n_A times, then the probability of event A is defined as "Relative frequency"
- **Relative Frequency Definition**:

The probability of an event A is defined as

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

Joint probability

- ✓ Joint probability is defined as the probability of joint (or) simultaneous occurrence of two (or) more events.
- ✓ Let A and B taking place, and is denoted by P (AB) or P(A∩B)

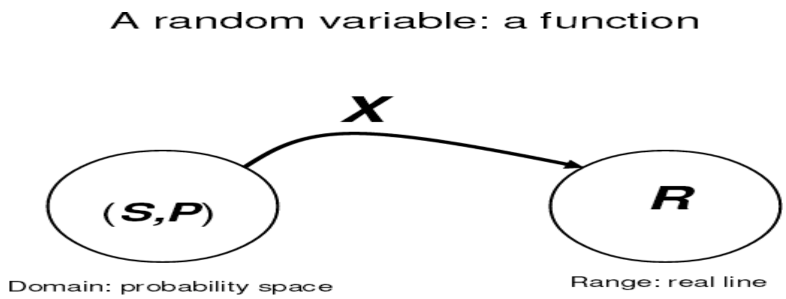
P (AB) = P(A∩B = P(A) + P(B) - P(AUB)

Random Variable

- ✓ A random variable is a real valued function that maps all the elements of sample space on to points on the real axis.

(OR)

- ✓ A random variable is a function that maps outcomes of a random experiment to real numbers.
- ✓ A (real-valued) random variable, often denoted by X(or some other capital letter), is a function mapping a probability space (S; P) into the real line R.
- ✓ The Figure Associated with each point s in the domain S the function X assigns one and only one value X(s) in the range R.



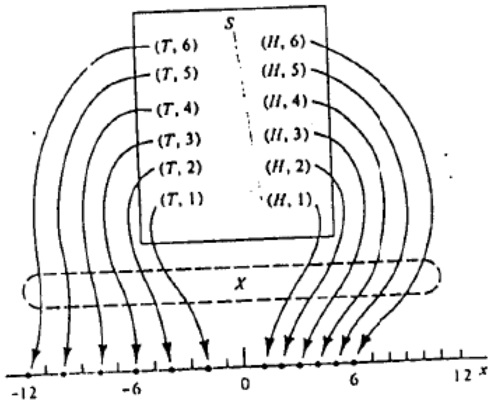
Classification of Random Variables

- ✓ Discrete Random Variable
- ✓ Continuous Random Variable
- ✓ Mixed Random Variable

Discrete Random Variable

A discrete random variable is a real valued function that maps all the elements of discrete or continuous sample space onto discrete points on the real axis.

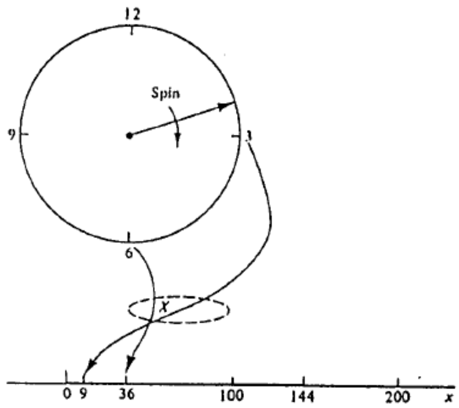
Example: Tossing coin



Continuous Random Variable

A continuous random variable is a real valued function that maps all the elements of continuous sample space onto continuous points on the real axis.

Example: Number on a chance of wheel



Mixed Random Variable

A mixed random variable is a real valued function that maps all the elements of continuous sample space onto either continuous points or discrete points on the real axis.

Conditions for a function to be a Random Variable

- ✓ Every point in sample space must be correspond only one value of the random variable.
- ✓ The set $\{X \leq x\}$ shall be an event for any real number x.The probability of this event is equal to the sum of the probabilities of all the elementary events corresponding to$\{X \leq x\}$. This is denoted as P$\{X \leq x\}$
- ✓ The probability of events $\{X \leq -\infty\}$ and $\{X \leq \infty\}$ are zero.

PROBABILITY DISTRIBUTION FUNCTION

- ✓ Probability distribution function $F_X(x)$ gives information about the probabilistic behaviour of a random variable.
- ✓ It gives information about probability of event $\{X \leq x\}$

$$F_X(x) = \mathrm{P}\{X \leq x\}$$

- ✓ Probability distribution function is also called as Cumulative distribution function (CDF).

Properties of Probability Distribution Function

1. The Probability distribution function is zero at

$$x = -\infty \qquad i.e;\ F_X(-\infty) = 0$$

Proof:

It is known that

$$F_X(x) = \mathrm{P}\{X \leq x\}$$

$$F_X(-\infty) = \mathrm{P}\{X \leq -\infty\}$$

$$F_X(-\infty) = 0$$

2. The Probability distribution function is unity at $x = \infty$

$$i.e;\ F_X(\infty) = 1$$

Proof:

It is known that

$$F_X(x) = \mathrm{P}\{X \leq x\}$$

$$F_X(\infty) = \mathrm{P}\{X \leq \infty\}$$

$$F_X(\infty) = \mathrm{P}\{X \leq -\infty\} + \cdots + \mathrm{P}\{X \leq 0\} + \mathrm{P}\{X \leq 1\} + \cdots \mathrm{P}\{X \leq \infty\}$$

$$F_X(\infty) = 1$$

3. The Probability distribution function is always define between 0 and 1.

   i.e; $0 \leq F_X(x) \leq 1$

4. The Probability distribution function is non decreasing function

   i.e; $F_X(x_2) \geq F_X(x_1)\ when\ x_2 > x_1$

5. The Probability of an event $x_1 < X \leq x_2$ can be obtained from the knowledge of Probability distribution function

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$$

Proof

Given that $x_2 > x_1$

$$\{X \leq x_2\} = \{X \leq x_1\} + \{x_1 \leq X \leq x_2\}$$

Adding probabilities to the above events

$$P\{X \leq x_2\} = P\{X \leq x_1\} + P\{x_1 \leq X \leq x_2\}$$

$$P\{x_1 \leq X \leq x_2\} = P\{X \leq x_2\} - P\{X \leq x_1\}$$

$$P\{x_1 \leq X \leq x_2\} = F_X(x_2) - F_X(x_1)$$

Probability Density Function:

- ✓ Probability density function will give the information about the probability of individual outcomes P$\{X = x\}$
- ✓ Let *F*(*x*) be the distribution function for a continuous random variable *X*. The **probability density function** (PDF) for *X* is given by

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Properties of Probability density function

1. Density function is a non-negative quantity

$$f_X(x) \geq o$$

Proof

From the definition $f_X(x) = \frac{dF_X(x)}{dx}$

As the distribution function is anon non-decreasing function slope is always positive. Hence the probability density function is a non-negative quantity.

2. The area under the probability density function is unity.

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Proof:

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)$$

$$[F_X(x)]_{-\infty}^{\infty} = F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0 = 1$$

3. The probability distribution function can be obtained from the knowledge of density function. It means distribution function is the area under the density function.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Proof

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Integrating on both sides

$$\int_{-\infty}^{x} f_X(x)dx = \int_{-\infty}^{x} \frac{d}{dx} F_X(x)$$

$$= [F_X(x)]_{-\infty}^{x}$$

$$= F_X(x) - F_X(-\infty) = F_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

4. The Probability of an event $\{x_1 < X \le x_2\}$ can be obtained from the knowledge of Probability distribution function

$$P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)dx$$

Proof:

From Probability distribution function

$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$$

$$= \int_{-\infty}^{x_2} f_X(x)dx - \int_{-\infty}^{x_1} f_X(x)dx$$

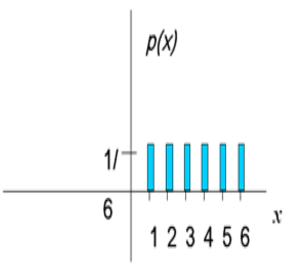
$$= \int_{-\infty}^{x_1} f_X(x)dx + \int_{x_1}^{x_2} f_X(x)dx + \int_{x_2}^{\infty} f_X(x)dx - \int_{-\infty}^{x_1} f_X(x)dx$$

$$= \int_{x_1}^{x_2} f_X(x)dx$$

$$P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)dx$$

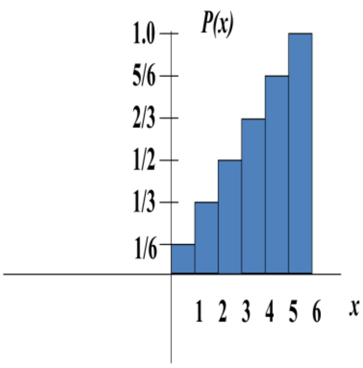
X	p(x)
1	p(x=1)=1/6
2	p(x=2)=1/6
3	p(x=3)=1/6
4	p(x=4)=1/6
5	p(x=5)=1/6
6	p(x=6)=1/6
	1.0

Probability Density Function (Pdf)



x	P(x≤A)
1	P(x≤1)=1/6
2	P(x≤2)=2/6
3	P(x≤3)=3/6
4	P(x≤4)=4/6
5	P(x≤5)=5/6
6	P(x≤6)=6/6

Cumulative Distribution Function (CDF)



Binomial Density and Distribution Function

This function is used in those applications where only two outcomes are possible.

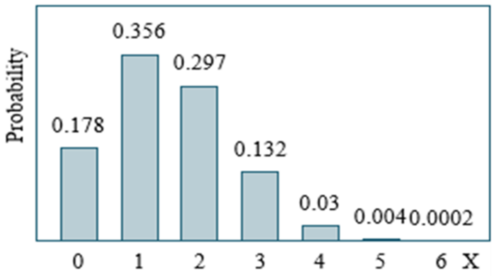
Probability density function

$$f_X(x) = \sum_{k=1}^{N} N_{C_k}\, P^k\, (1-P)^{N-k}\ \delta(x-k)$$

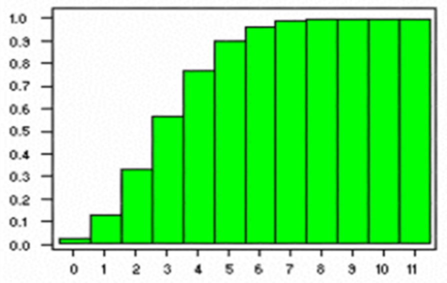
Probability distribution function

$$F_X(x) = \sum_{k=1}^{N} N_{C_k}\, P^k\, (1-P)^{N-k}\ u(x-k)$$

where P – Probability of occurrence of event



density function



distribution function

Poisson Density and Distribution Function

Probability density function

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!}\ \delta(x-k)$$

where b>0 is a real constant

$$b = \lambda\, T$$

$$\lambda = Average\ rate \quad T = time\ interval$$

Probability distribution function

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!}\ u(x-k)$$

Applications

It is mostly applied to counting type problems

- ✓ The no.of telephone calls made during a period of time
- ✓ The no.of defective elements in a given samples
- ✓ The no.of items waiting in a queue

Gaussian Density and Distribution Function

This is the most prominently used density function in the field of science and engineering.

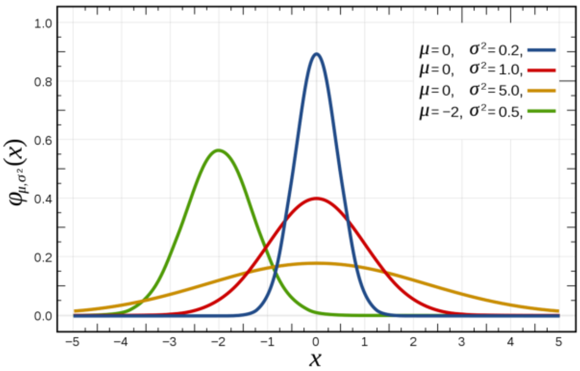
In particular Gaussian function is used to approximate noise in communication related applications.

Gaussian density function

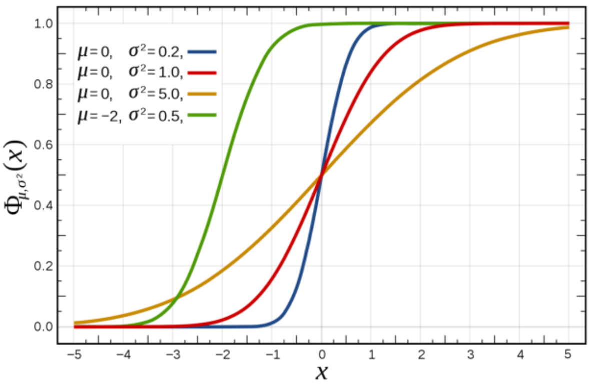
$$f_X(x) = \frac{1}{\sqrt{2\pi{\sigma_x}^2}}\ e^{-\left(\frac{(x-\mu_x)^2}{2{\sigma_x}^2}\right)}$$

Where $\mu_{x\ and}\ {\sigma_x}^2$ are mean and variance of random variable X.

σ_x is standard deviation.



Gaussian density function



Gaussian distribution function

Uniform Density and Distribution Function

This function is used to describe the behaviour of the random variable that is same for a given time interval.

Uniform density function

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le X \le b \\ 0 & else\ where \end{cases}$$

Uniform distribution function

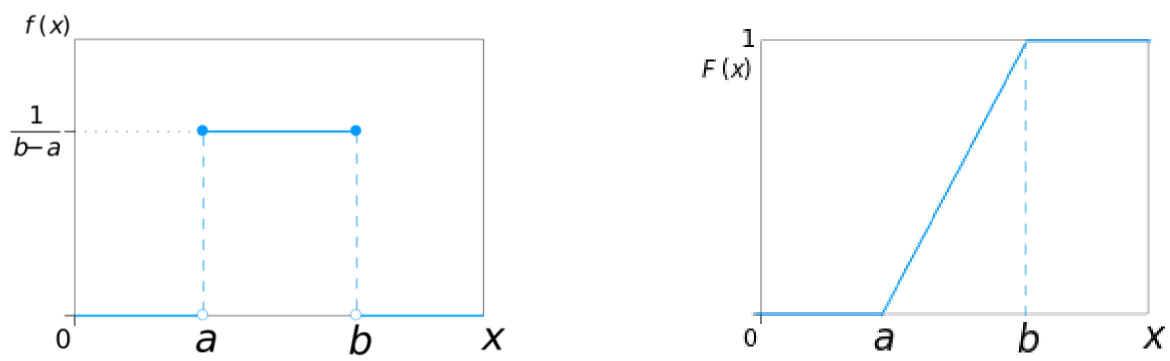
$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$= \int_{a}^{x} \frac{1}{b-a} \quad dx \ = \ \frac{1}{b-a} \ x]_a^x = \frac{x-a}{b-a}$$

$$F_X(a) = \ 0$$

$$F_X(b) = \frac{b-a}{b-a} = 1$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le X < b \\ 1 & x \ge b \end{cases}$$



a) Density function b) Distribution function

Applications

- ✓ *The random distribution of errors introduced in the round off process is uniformly distributed.*
- ✓ *In digital communications during sampling process.*

Exponential Density and Distribution Function

Probability density function of a random variable X is defined as

$$f_X(x) = \begin{cases} \frac{1}{b}\, e^{-\left(\frac{x-a}{b}\right)} & x > a \\ 0 & x < a \end{cases}$$

where a and b are real constants $-\infty < a < \infty$ *and* $b > 0$

Probability distribution function

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b}\, e^{-\left(\frac{x-a}{b}\right)}dx$$

$$F_X(x) = \int_{-\infty}^{a} \frac{1}{b}\, e^{-\left(\frac{x-a}{b}\right)}dx + \int_{a}^{x} \frac{1}{b}\, e^{-\left(\frac{x-a}{b}\right)}dx$$

$$F_X(x) = 1 - e^{-\left(\frac{x-a}{b}\right)}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\left(\frac{x-a}{b}\right)} & x > a \\ 1 & x = \infty \end{cases}$$

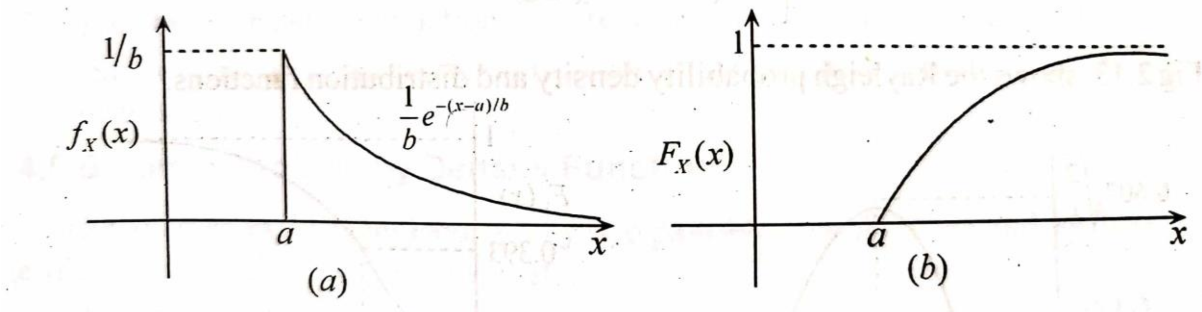


Fig. 2.12: a) Exponential density function b) Exponential distribution function

Applications

- ✓ The distribution of fluctuations in signal strength received by radar receivers from certain types of targets.
- ✓ The distribution of raindrop sizes when a large number of rainstorm measurements are made.

Rayleigh Density and Distribution Function

Probability density function of a random variable X is defined as

$$f_X(x) = \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} \qquad x \geq a$$

$$0 \qquad x < a$$

where a and b are real constants $-\infty < a < \infty \; and \; b > 0$

Probability distribution function

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

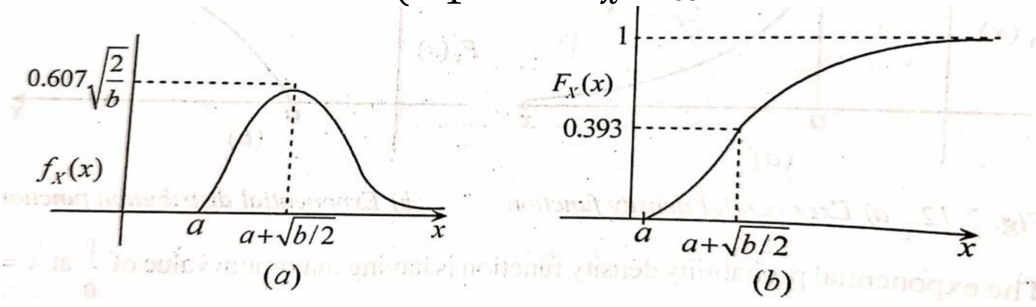
$$F_X(x) = \int_{-\infty}^{x} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

$$\text{Let } \frac{(x-a)^2}{b} = y \implies \frac{2}{b}(x-a)dx = dy$$

$$F_X(x) = \int_{a}^{x} e^{-y}dy$$

$$= -e^{-y}\Big]_a^x = -e^{-\frac{(x-a)^2}{b}}\Bigg]_a^x = 1 - e^{-\frac{(x-a)^2}{b}}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 1 & x = \infty \end{cases}$$



a) Density function b) Distribution function

Applications

- ✓ It describes the envelope of white noise, when the noise is passed through a band pass filter.
- ✓ Some types of signal fluctuations received by receivers are modelled as Rayleigh distribution

OPERATIONS ON RANDOM VARIABLES

- ✓ Various operations on random variable will indicate the way it behaves.
- ✓ These operations are used to perform statistical analysis of real time experiment where they exists much randomness.
- ✓ Expected values as well as moments are the parameter is used for describing the behaviour of random variable.

Expected Value or Mean

Expectation is the fundamental operation used in the statistical analysis.

If X is a continuous RV with valid pdf $f_X(x)$, then the expected value of X or the mean value of X is defined as

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x * f_X(x)\ dx$$

If X is a discrete RV

$$E[X] = \bar{X} = \sum_{i=1}^{N} x_i\ P(x_i)$$

Expected Value of a function of a Random Variable

If X is a continuous RV with valid pdf $f_X(x)$, then the expected value of function g(x) is defined as

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

If X is a discrete RV, then the expected value of function g(x) is defined as

$$E[X] = \bar{X} = \sum_{i=1}^{N} g(x_i)\ \ (P(x_i)$$

Properties of Expectations

1. The expected value of a constant is constant.

Proof

$$E[X] = \int_{-\infty}^{\infty} x. f_X(x)\ dx$$

$$E[k] = \int_{-\infty}^{\infty} k. f_X(x)\ dx$$

$$E[X] = k. \int_{-\infty}^{\infty} f_X(x)\ dx$$

$E[X] = k. 1$

$E[X] = k$

2. Let $E[X]$ be the expected value of a RV 'X' then

   $E[aX] = a\ E[X]$

Proof

$$E[X] = \int_{-\infty}^{\infty} x. f_X(x)\ dx$$

$$E[aX] = \int_{-\infty}^{\infty} ax. f_X(x)\ dx$$

$$= a \int_{-\infty}^{\infty} x. f_X(x)\ dx$$

$E[aX] \ \ = a\ E[X]$

3. Let $E[X]$ be the expected value of a RV 'X' then

   $E[aX + b] = aE[X] + b$

$$E[X] = \int_{-\infty}^{\infty} x. f_X(x)\ dx$$

$$E[aX + b] = \int_{-\infty}^{\infty} (ax + b). f_X(x)\ dx$$

$$= a \int_{-\infty}^{\infty} x. f_X(x)\ dx + b. \int_{-\infty}^{\infty} f_X(x)\ dx$$

$E[aX] \ \ = aE[X]$+b

Moments:

Moment of a RV describes the deviation from a reference value.

- ✓ ***Moments about the origin***
- ✓ ***Moments about the Mean***

Moments about the origin

The expected value of a given function $g(x) = X^n$

is called n[th] moment about the origin.

$$m_n = E(X^n)$$

$$= \int_{-\infty}^{\infty} x^n . f_X(x)\ dx$$

$$m_1 = E(X) = \int_{-\infty}^{\infty} x . f_X(x)\ dx$$

The first moment about the origin is nothing but mean value (or) expected value of a random variable.

Second Moment

$$m_2 = E(X^2) = \int_{-\infty}^{\infty} x^2 . f_X(x)\ dx$$

Moments about the Mean

The expected value of a given function $g(x) = (X - \bar{X})^n$ is called n[th] central moment (or) n[th] moment about the mean.

$$\mu_n = E(X - \bar{X})^n$$

$$= \int_{-\infty}^{\infty} (x - \bar{X})^n . f_X(x)\ dx$$

$\mu_0 = 1$

$\mu_1 = \int_{-\infty}^{\infty} x . f_X(x)\ dx - \int_{-\infty}^{\infty} \bar{X} . f_X(x)\ dx$

$= \bar{X} - \bar{X}$

$= 0$

Variance:

The second central moment of a random variable 'X' is called Variance.

It is defined as the expected value of a function of the form $g(x) = (X - \bar{X})^2$

where $\bar{X} - mean\ value.$

$$\mu_2 = {\sigma_X}^2 = var(x)$$

The variance is used to calculate the average power of a random signal in communication related applications.

Relation between variance and moments about the origin

$${\sigma_X}^2 = E(X - \bar{X})^2 = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2 + \bar{X}^2 - 2x\bar{X})\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx + (\bar{X})^2 \int_{-\infty}^{\infty} f_X(x)\,dx - 2\bar{X} \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= m_2 + \bar{X}^2 - 2\,\bar{X}\,\bar{X}$$

$$= m_2 + {m_1}^2 - 2\,{m_1}^2$$

$${\sigma_X}^2 = m_2 - {m_1}^2$$

Properties of Variance:

1. The variance of a constant is zero.

$$var[k] = 0$$

Proof:

It is known that $var[X] = E(X - \bar{X})^2$

$$var[k] = E\left(k - \bar{k}\right)^2$$

as k is constant $k = \bar{k}$

$$var[k] = E(k - k)^2$$

$$var[k] = 0$$

2 Let $var[X]$ denote the variance of a RV 'X' then

$$var[aX] = a^2 var[X]$$

Proof:

$$var[X] = E(X - \bar{X})^2$$

$$\begin{aligned} var[aX] &= E(aX - a\bar{X})^2 \\ &= E[a^2(X - \bar{X})^2] \\ &= a^2 E(X - \bar{X})^2 \\ &= a^2 var[X] \end{aligned}$$

3. Let $var[X]$ denote the variance of a RV 'X' then

$$var[aX + b] = a^2 var[X]$$

Proof:

$$var[X] = E(X - \bar{X})^2$$

$$\begin{aligned} var[aX + b] &= E\left((aX + b) - \left(\overline{aX + b}\right)\right)^2 \\ &= E\left((aX + b) - (a\bar{X} + b)\right)^2 \\ &= E(aX - a\bar{X})^2 \\ &= E[a^2(X - \bar{X})^2] \\ &= a^2 E(X - \bar{X})^2 \\ &= a^2 var[X] \end{aligned}$$

SKEWNESS:

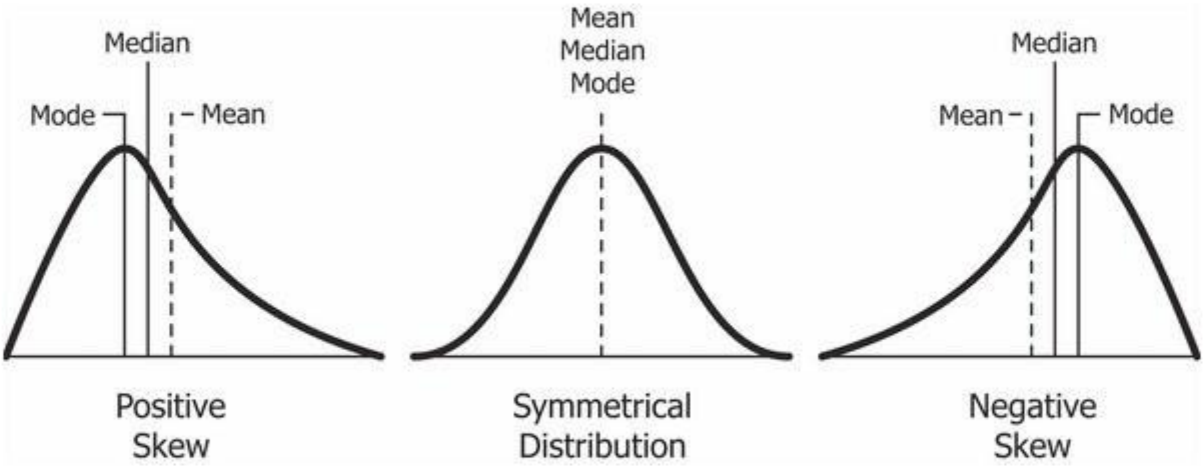
The third central moment of a random variable 'X' is called skewness.

It gives the asymmetry of a random variable with reference to a given value.

$$\mu_3 = E(X - \bar{X})^3 = \int_{-\infty}^{\infty} (X - \bar{X})^3 f_X(x)\, dx$$

It is the *degree of distortion* from the symmetrical bell curve or the normal distribution. It measures the lack of symmetry in data distribution. It differentiates extreme values in one versus the other tail. A symmetrical distribution will have a skewness of 0.

There are two types of Skewness: Positive and Negative



Positive Skewness means when the tail on the right side of the distribution is longer or fatter. The mean and median will be greater than the mode.

Negative Skewness is when the tail of the left side of the distribution is longer or fatter than the tail on the right side. The mean and median will be less than the mode.

Note:

- If the skewness is between -0.5 and 0.5, the data are fairly symmetrical.
- If the skewness is between -1 and -0.5(negatively skewed) or between 0.5 and 1(positively skewed), the data are moderately skewed.
- If the skewness is less than -1(negatively skewed) or greater than 1(positively skewed), the data are highly skewed.

Coefficient of skewness:

It is defined as the ratio of 3rd central moment to cube of standard deviation.

$$= \frac{\mu_3}{{\sigma_X}^3}$$

FUNCTIONS THAT GIVE MOMENTS

Along with probability density function two more functions are also used to describe the behaviour of a random variable. They are

- ✓ CHARACTERISTICS FUNCTION
- ✓ MOMENT GENERATING FUNCTIONS

These two functions are useful for calculating the n^{th} moments of a random variable.

CHARACTERISTICS FUNCTION

Consider a random variable X with pdf $f_X(x)$, then the expected value of the function $e^{j\omega X}$ is called characteristics function.

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

It is a function of real variable $-\infty < \omega < \infty$

where j is an imaginary operator

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega X} f_X(x)\,\mathrm{dx}$$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega X} \phi_x(\omega)\,\mathrm{dw}$$

$\phi_x(\omega)$ *and* $f_X(x)$ are Fourier transform pairs with the sign of the variable is reversed.

Properties of characteristics function

1. The value of characteristic function is unity at origin, i.e; ω =0

$$\phi_X(0) = 1$$

Proof

It is known that

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

$$\phi_X(0) = E\left[e^{j\omega 0}\right] = E[1] = 1$$

2. The maximum amplitude of characteristics function is unity at origin,

$$|\phi_X(\omega)| \leq 1$$

Proof

It is known that

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

The amplitude of $\phi_x(\omega)$ is

$$|\phi_X(\omega)| = \left|E\left[e^{j\omega X}\right]\right|$$

$$= \left|\int_{-\infty}^{\infty} e^{j\omega X} f_X(x)\,\mathrm{dx}\right|$$

Since $|XY| \leq |X|\,|Y|$

$$|\phi_X(\omega)| \leq \int_{-\infty}^{\infty} \left|e^{j\omega X}\right| |f_X(x)|\,\mathrm{dx}$$

$$since \left|e^{j\omega X}\right| = 1$$

$$|\phi_X(\omega)| \leq \int_{-\infty}^{\infty} |f_X(x)|\,\mathrm{dx}$$

$$|\phi_X(\omega)| \leq 1$$

3. If $\phi_x(\omega)$ be the characteristic function of a random variable X, then characteristic function of $Y = aX + b$ is given by $\phi_Y(\omega) = e^{j\omega b}\phi_x(a\omega)$

Proof:

$$\text{Given } Y = aX + b$$

$$\phi_Y(\omega) = E\left[e^{j\omega Y}\right]$$

$$= E\left[e^{j\omega(aX+b)}\right]$$

$$= E\left[e^{j\omega aX}\, e^{j\omega b}\right]$$

$$= e^{j\omega b}\, E\left[e^{j\omega aX}\right]$$

$$\phi_Y(\omega) = e^{j\omega b}\phi_X(a\omega)$$

4. If X_1 and X_2 are two independent random variables then

$$\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega) * \phi_{X_2}(\omega)$$

Proof

It is known that

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

$$\phi_{X_1+X_2}(\omega) = E\left[e^{j\omega(X_1+X_2)}\right]$$

$$= E\left[e^{j\omega X_1}\, e^{j\omega X_2}\right]$$

$$= E\left[e^{j\omega X_1}\right] E\left[e^{j\omega X_2}\right]$$

Since X_1 and X_2 are two independent random variables

$$\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega) * \phi_{X_2}(\omega)$$

5. The n[th] moment of random variable can be obtained from the knowledge of characteristic function is

$$m_n = (-j)^n \left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

Proof:

Consider $$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, \mathrm{dx}$$

Differentiate both side w.r.to ω

$$\frac{d^n\phi_X(\omega)}{d\omega^n} = \frac{d^n}{d\omega^n}\int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, \mathrm{dx}$$

Changing the order

$$\frac{d^n\phi_X(\omega)}{d\omega^n} = \int_{-\infty}^{\infty} \frac{d^n}{d\omega^n} e^{j\omega x} f_X(x)\, \mathrm{dx}$$

$$= \int_{-\infty}^{\infty} (jx)^n e^{j\omega x} f_X(x)\, \mathrm{dx}$$

$$= (j)^n \int_{-\infty}^{\infty} (x)^n e^{j\omega x} f_X(x)\, \mathrm{dx}$$

$$\left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0} = (j)^n \int_{-\infty}^{\infty} (x)^n \; f_X(x)\,\mathrm{dx}$$

$$= (j)^n \; m_n$$

$$m_n = (-j)^n \left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

MOMENT GENERATING FUNCTIONS

The moment generating function of a random variable is also used to generate the n^{th} moment about the origin.

Consider a random variable X with pdf $f_X(x)$, then the expected value of the function $e^{\nu X}$ is called moment generating function

$$M_X(\nu) = E[e^{\nu X}]$$

where ν is a real variable $-\infty < \nu < \infty$

$$M_X(\nu) = \int_{-\infty}^{\infty} e^{\nu X} \, f_X(x)\,\mathrm{dx}$$

Properties of moment generating function:

1. The moment generating function is unity at origin

$$M_X(0) = 1$$

Proof:

$$M_X(\nu) = \int_{-\infty}^{\infty} e^{\nu X} \, f_X(x)\,\mathrm{dx}$$

$$M_X(0) = \int_{-\infty}^{\infty} e^{0.X} \, f_X(x)\,\mathrm{dx}$$

$$M_X(0) = \int_{-\infty}^{\infty} f_X(x)\,\mathrm{dx}$$

$$M_X(0) = 1$$

2. Let $M_X(\nu)$ be the moment generating function of a random variable X, then moment generating function of $Y = aX + b$ is given by

$$M_Y(\nu) = e^{\nu b} M_X(\nu)$$

Proof:

$$M_Y(\nu) = E[e^{\nu Y}]$$

$$= E\left[e^{\nu(aX+b)}\right]$$

$$= E[e^{\nu aX} \quad e^{\nu b}]$$

$$= e^{\nu b}\ E[e^{\nu aX}\ ]$$

$$M_Y(\nu) = e^{\nu b} M_X(\nu)$$

3. The n^th moment of random variable can be obtained from the knowledge of moment generating function is

$$m_n = \left.\frac{d^n M_X(\nu)}{d\nu^n}\right|_{\nu=0}$$

Proof:

Differentiate both side w.r.to ν

$$\frac{d^n M_X(\nu)}{d\nu^n} = \frac{d^n}{d\nu^n}\int_{-\infty}^{\infty} e^{\nu x}\ f_X(x)\ \mathrm{dx}$$

Changing the order

$$= \int_{-\infty}^{\infty} \frac{d^n}{d\nu^n} e^{\nu x}\ f_X(x)\ \mathrm{dx}$$

$$= \int_{-\infty}^{\infty} (x)^n e^{\nu x}\ f_X(x)\ \mathrm{dx}$$

$$= \int_{-\infty}^{\infty} (x)^n e^{\nu x}\ f_X(x)\ \mathrm{dx}$$

$$\left.\frac{d^n M_X(\nu)}{d\nu^n}\right|_{\nu=0} = \int_{-\infty}^{\infty} (x)^n\ \ f_X(x)\ \mathrm{dx} \quad = m_n$$

$$m_n = \left.\frac{d^n M_X(\nu)}{d\nu^n}\right|_{\nu=0}$$

TRANSFORMATION ON RANDOM VARIABLES

In majority of the real time applications random variable is to be converted into another random variable based on certain type of transformation.

Let 'X' be a random variable then $Y = T(X)$

where T is the operation (or) transformation performed on 'X'.

- ✓ The nature of Y depends on the type of X as well as type of transformation is used.
- ✓ The random variable X is either continuous (or) discrete (or) mixed. Based on this number of transmissions exists like linear, non-linear, segmentation, stair case, etc

The following are the specific cases of transformations

1. Both X and T are continuous and T is either Monotonic Increasing or Decreasing with X
2. Both X and T are continuous and T is Non-Monotonic
3. X is discrete and T is continuous.

Monotonic Transformation of a Continuous Random Variable

- A function is said to be monotonically increasing if $T(X_2) > T(X_1)$ for any $x_1 < x_2$
- A function is said to be monotonically decreasing if $T(X_2) < T(X_1)$ for any $x_1 < x_2$

Monotonic Increasing Transformation

For transforming a random variable X into Y the distribution function of both X and y must be same.

Let there exist a random variable X such that $x = x_0$ then $y_0 = T(x_0)$ (or) $x_0 = T^{-1}(y_0)$

As the distribution function should be same

$$\mathrm{P}\{Y \leq y_0\} = \mathrm{P}\{X \leq x_0\}$$

$$F_Y(y_0) = F_X(x_0)$$

We know that $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

$$\int_{-\infty}^{y_0} f_Y(y)dy = \int_{-\infty}^{x_0} f_X(x)dx$$

Differentiating on both sides w.r.to 'y'

$$\frac{d}{dy}\int_{-\infty}^{y_0} f_Y(y)dy = \frac{d}{dy}\int_{-\infty}^{x_0} f_X(x)dx$$

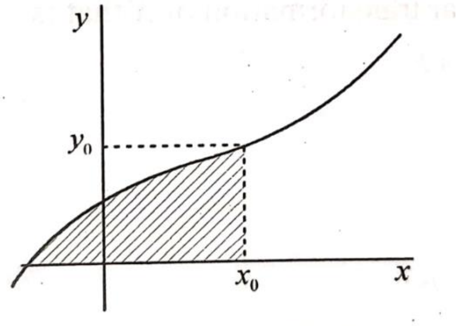
Using Leibniz rule

$$f_Y(y_0) = f_X(x_0)\frac{dx_0}{dy}$$

$$f_Y(y_0) = f_X(T^{-1}(y_0))\frac{d}{dy}(T^{-1}(y_0))$$

The above transformation is applicable for every value of y and x resulting in

$$f_Y(y) = f_X(x)\frac{dx}{dy}$$



Monotonic increase function

Monotonic Decreasing Transformation

$$\{Y \le y_0\} = \{X \ge x_0\}$$

$$\mathrm{P}\{Y \le y_0\} = \mathrm{P}\{X \ge x_0\}$$

$$\mathrm{P}\{Y \le y_0\} = 1 - \mathrm{P}\{X < x_0\}$$

$$F_Y(y_0) = 1 - F_X(x_0)$$

We know that $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

$$\int_{-\infty}^{y_0} f_Y(y)dy = 1 - \int_{-\infty}^{x_0} f_X(x)dx$$

Differentiating on both sides w.r.to 'y'

$$\frac{d}{dy}\int_{-\infty}^{y_0} f_Y(y)dy = \frac{d}{dy}\left[1 - \int_{-\infty}^{x_0} f_X(x)dx\right]$$

Using Leibniz rule

$$f_Y(y_0) = -f_X(x_0)\frac{dx_0}{dy}$$

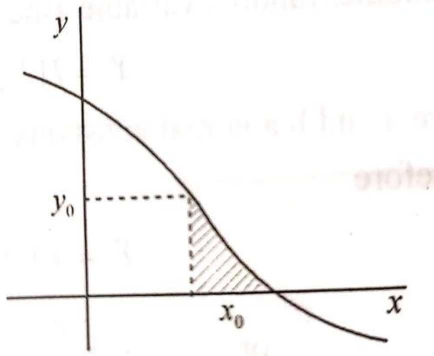
$$f_Y(y_0) = -f_X(T^{-1}(y_0))\frac{d}{dy}(T^{-1}(y_0))$$

The above transformation is applicable for every value of y and x resulting in

$$f_Y(y) = -f_X(x)\frac{dx}{dy}$$

For monotonic transformation either increasing or decreasing, the density function of y is

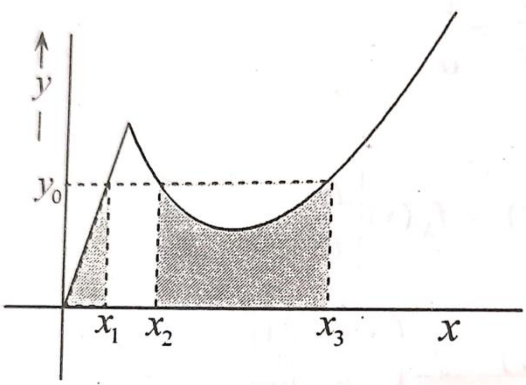
$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$



Monotonic decreasing function

Non-Monotonic Transformation of a Continuous Random Variable

Consider a random variable Y is a Non-Monotonic Transformation of a Continuous Random Variable X as shown in figure



Non monotonic function y

For a given event $\{Y \leq y_0\}$, there are more than one value of X.

It is observed that the event $\{Y \leq y_0\}$ corresponds to the events $\{X \leq x_1 and\ x_2 < X \leq x_3\}$

Thus the probability of the event $\{Y \leq y_0\}$ is equal to the probability of the event $\{X/Y \leq y_0\}$

$$\mathrm{P}\{Y \leq y_0\} = \mathrm{P}\{X/Y \leq y_0\}$$

$$F_Y(y_0) = \int_{X/Y \leq y_0} f_X(x)dx$$

By differentiating, the density function is given by

$$\frac{d}{dy} F_Y(y_0) = \frac{d}{dy} \int_{X/Y \leq y_0} f_X(x)dx$$

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + f_X(x_3)\left|\frac{dx_3}{dy}\right| \ldots.$$

MEAN AND VARIANCE FOR POISSON RANDOM VARIABLE

We know that the Probability density function for Poisson random variable is

$$f_X(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\, \delta(x-k)$$

The probability function is

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad x = 0,1,2,3 \ldots \ldots$$

Expected Value

If X is a discrete RV

$$E[X] = \bar{X} = \sum_{i=1}^{N} x_i\, P(x_i)$$

$$E[X] = \bar{X} = \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x(x-1)!}$$

$$= \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x(x-1)!}$$

$$= \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda\ \lambda^{x-1}}{(x-1)!}$$

$$= \lambda\, e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda\, e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots . \right)$$

$$= \lambda\, e^{-\lambda}\ e^{\lambda}$$

$$E[X] = m_1 = \lambda$$

VARIANCE:

$$\sigma_X{}^2 = m_2 - m_1{}^2$$

$$E(X^2) = m_2 = \sum_{i=1}^{N} x_i{}^2\, P(x_i)$$

$$m_2 = \sum_{x=0}^{\infty} x^2\, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$x^2 = x(x-1) + x$$

$$= \sum_{x=0}^{\infty} (x(x-1) + x)\, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\, e^{-\lambda} \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\, e^{-\lambda} \frac{\lambda^x}{x(x-1)(x-2)!} + \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x(x-1)!}$$

$$= \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^2\, \lambda^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda\, \lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2\, e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda\, e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2\, e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots . \right) + \lambda\, e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots . \right)$$

$$= \lambda^2 e^{-\lambda}\ e^{\lambda} + \lambda\, e^{-\lambda}\ e^{\lambda}$$

$$= \lambda^2 + \lambda$$

Variance

$$\sigma^2 = m_2 - {m_1}^2$$
$$\sigma^2 = \lambda^2 + \lambda - \lambda^2$$
$$\sigma^2 = \lambda$$

For the Poisson random variable

$$E[X] = \boldsymbol{\sigma^2} = \lambda$$

MEAN AND VARIANCE FOR UNIFORM RANDOM VARIABLE

We know that the Probability density function for uniform random variable is

$$f_X(x) = \frac{1}{b-a} \qquad a \le X \le b$$
$$= 0 \qquad othewise$$

Expected Value

$$m_1 = E[X] = \bar{X} = \int_{-\infty}^{\infty} x\, *\, f_X(x)\ dx$$

$$= \int_a^b x\, \frac{1}{b-a} \quad dx$$

$$= \frac{1}{b-a} \int_a^b x\ \ dx$$

$$= \frac{1}{b-a} \frac{x^2}{2}\bigg]_a^b$$

$$= \frac{1}{2} \frac{b^2 - a^2}{b-a}$$

$$= \frac{1}{2} \frac{(b-a)(b+a)}{b-a}$$

$$m_1 = E[X] = \frac{(b+a)}{2}$$

VARIANCE:

$$\sigma_X{}^2 = m_2 - m_1{}^2$$

$$m_2 = E(X^2) = \int_{-\infty}^{\infty} x^2 . f_X(x)\ dx$$

$$= \int_a^b x^2 \frac{1}{b-a} \quad dx$$

$$= \frac{1}{b-a} \int_a^b x^2\ dx$$

$$= \frac{1}{b-a} \frac{x^3}{3}\bigg]_a^b$$

$$= \frac{1}{3} \frac{b^3 - a^3}{b-a}$$

$$= \frac{1}{3} \frac{(b-a)(b^2 + a^2 + ab)}{b-a}$$

$$m_2 = \frac{(b^2 + a^2 + ab)}{3}$$

Variance

$$\sigma^2 = m_2 - m_1{}^2$$

$$\sigma^2 = \frac{(b^2 + a^2 + ab)}{3} - \left(\frac{b+a}{2}\right)^2$$

$$\sigma^2 = \frac{(b^2 + a^2 + ab)}{3} - \frac{(b^2 + a^2 + 2ab)}{4}$$

$$\sigma^2 = \frac{(b^2 + a^2 - 2ab)}{12}$$

$$\sigma^2 = \frac{(b - a)^2}{12}$$

For the Uniform random variable

$$E[X] = \frac{(b + a)}{2}$$

$$\sigma^2 = \frac{(b - a)^2}{12}$$

Descriptive Questions

1. Summarize the properties of Probability distribution function with relevant proofs.
2. Discuss the properties of characteristic function with the help of necessary expressions.
3. Contrast the properties of Moment Generating Function.
4. Describe the properties of probability density Function.
5. A random variable X has a probability density

$$f_X(x) = \begin{cases} \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right), & -4 < x < 4 \\ 0, & elsewhere \end{cases}$$

Find (a) Mean value (b) Second moment (c) Variance.

6. A random variable X has the density function

$$f_X(x) = \begin{cases} k\left(\frac{1}{1+x^2}\right), & -\infty < x < \infty \\ 0, & elsewhere \end{cases}$$

Find k, $F_X(x)$ and $P(X \geq 0)$

7. A random variable has a probability density

$$f_X(x) = \begin{cases} \frac{5}{4}\left(1-x^4\right), & 0 < x \leq 1 \\ 0, & elsewhere \end{cases}$$

Calculate (a) E[X] (b) E[4X+2] (c) E[X^2].

8. A random variable X has $\overline{X} = -3, \overline{X^2} = 11, and \sigma_x^2 = 2.$ For a new random variable Y= 2X-3, find $(i)\overline{Y}, (ii)\overline{Y^2}, and (iii)\sigma_y^2.$

9. The probability density function of a random variable X is given as f_X (x)=ae^{-bx} for x ≥0.
   Find (i)Moment generating Function (ii) E[X]

10. A random variable X has the density

$$f_X(x) = \begin{cases} \frac{3}{32}\left(-x^2+8x-12\right), & 2 \leq x \leq 6 \\ 0, & elsewhere \end{cases}$$

Evaluate the Mean and Variance of X

PROBLEMS

1. Find the constant 'b' so that the given density function is a valid function.

$$f_X(x) = \begin{cases} e^{3x/4}, & 0 < x < b \\ 0, & else\ where \end{cases}$$

Sol:

$$we\ know\ that \int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$\int_0^b e^{3x/4} dx = 1$$

$$\left[\frac{e^{3x/4}}{\frac{3}{4}}\right]_0^b = 1$$

$$\left[e^{\frac{3b}{4}} - 1\right] = \frac{3}{4}$$

$$e^{\frac{3b}{4}} = \frac{7}{4}$$

$$\frac{3b}{4} = \ln\left(\frac{7}{4}\right)$$

b= 0.7461

2. Assume automobile arrivals at a gasoline station are Poisson and occurs at an average rate of 50per/Hour. The station has only one gasoline pump. If all cars are assumed to require one minute to obtain fuel, What is the probability that a weighting line will occur at the pump?

Sol:

If two or more cars arrive, then weighting line will occur at the pump

$$P(X \geq 2) = 1 - P(X \leq 1)$$

$$= 1 - F_X(1)$$

$$\text{Poisson distribution } F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!}$$

$$= 1 - e^{-b} \sum_{k=0}^{1} \frac{b^k}{k!}$$

$$b = \lambda\, T = \frac{50}{60} * 1 = 5/6$$

$$= 1 - e^{-5/6} \sum_{k=0}^{1} \frac{(\frac{5}{6})^k}{k!}$$

$$= 1 - e^{-\frac{5}{6}} * \frac{5}{6}$$

$$P(X \geq 2) = 0.203$$

3. A RV 'X' is having probability distribution function

$$F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} \mu(x-n).$$

Obtain a) $P(-\infty < X \leq 6)$ b) $P(X > 4)$ c) $P(6 < X \leq 9)$

Sol: $P(-\infty < X \leq 6) = \sum_{n=1}^{6} \frac{n^2}{650}$

$$= \frac{1}{650} \sum_{n=1}^{6} n^2$$

$$= \frac{1}{650} \left[\frac{n(n+1)(n+2)}{6}\right]_6$$

$$= 0.14$$

$b)P(X > 4) = 1 - P(X \leq 4)$

$$= 1 - \sum_{n=1}^{4} \frac{n^2}{650} = 0.9538$$

$P(6 < X \leq 9)$

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$$

$$= F_X(9) - F_X(6)$$

$$= \sum_{n=1}^{9} \frac{n^2}{650} - \sum_{n=1}^{6} \frac{n^2}{650} = 0.2984$$

4. A random variable X has the density function is given. Find k, $F_X(x)$ and $P(X \geq 0)$

$$f_X(x) = \begin{cases} k\left(\dfrac{1}{1+x^2}\right), & -\infty < x < \infty \\ 0, & elsewhere \end{cases}$$

Sol: We know that $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$k\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 1$$

$$k \tan^{-1} x]_{-\infty}^{\infty} = 1$$

$$k\left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right) = 1$$

$$k\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1 \Rightarrow k = \frac{1}{\pi}$$

ii) The distribution function $F_X(x)$

$$F_X(x) = k\int_{-\infty}^{x} \frac{1}{1+x^2} dx$$

$$= k \tan^{-1} x]_{-\infty}^{x}$$

$$= k\left(\tan^{-1}(x) - \tan^{-1}(-\infty)\right)$$

$$= \frac{1}{\pi}\left(\tan^{-1}(x) + \frac{\pi}{2}\right)$$

iii) $P(X \geq 0) = k\int_0^{\infty} \frac{1}{1+x^2} dx$

$$= k \tan^{-1} x]_0^{\infty}$$

$$= k\left(\tan^{-1}(\infty) - \tan^{-1}(0)\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 0.5$$

5. A discrete random variable X takes values from 1 to 5 with probabilities P(X) as given below. Compute the mean and variance of the random variable X.

X	1	2	3	4	5
P(X)	0.1	0.2	0.4	0.2	0.1

Sol:

$$mean\ \ m_1 = E[X] = \bar{X} = \sum_{i=1}^{N} x_i\ P(x_i)$$

$$= \sum_{i=1}^{5} x_i\ P(x_i)$$

$$\bar{X} = 1 * (0.1) + 2 * (0.2) + 3 * (0.3) + 4 * (0.2) + 5 * (0.1) = \mathbf{3}$$

$variance \quad {\sigma_X}^2 = m_2 - {m_1}^2$

$$m_2 = \sum_{i=1}^{N} {x_i}^2\ P(x_i)$$

$$m_2 = 1 * (0.1) + 4 * (0.2) + 9 * (0.3) + 16 * (0.2) + 25 * (0.1)$$

$$m_2 = 9.3$$

$$\begin{aligned} {\sigma_X}^2 &= m_2 - {m_1}^2 \\ &= 9.3 - 9 \\ &= \mathbf{0.3} \end{aligned}$$

6. A random variable X has $\overline{X} = -3, \overline{X^2} = 11, and\, \sigma_x^2 = 2.$ For a new random variable Y= 2X-3, find

$$i)\ \bar{Y} \qquad ii)\ \ \overline{Y^2} \qquad iii)\ \ {\sigma_Y}^2$$

Sol: Given Y= 2X-3

$$\bar{Y} = E[2X - 3]$$

$$= 2\ E[X] - 3$$

$$= 2\ (-3) - 3 = -9$$

$$\overline{Y^2} = E[(2X - 3)^2]$$

$$= E[4X^2 - 12X + 9]$$

$$= 4E[X^2] - 12E[X] + 9$$

$$= (4 * 11) - (12 * -3) + 9 = 89$$

$$\sigma_Y{}^2 = m_2 - m_1{}^2$$

$$\sigma_Y{}^2 = 89 - 81 = 8$$

(OR)

$$\sigma_Y{}^2 = var(Y)$$

$$= var(2X - 3)$$

Since the variance of a constant is zero

$var[aX] = a^2 var[X]$

$$= 4var(X)$$

$$= 4 * 2 = 8$$

7. Obtain (i)Moment generating Function (ii) E[X] of the probability density function of a random variable X is given as

$$f_X(x) = ae^{-x} \text{ for x } \geq 0.$$

Consider

$$M_X(v) = E[e^{vX}] = \int_0^\infty e^{vx}\, a\, e^{-x}\, \text{dx}$$

$$= a\int_0^\infty e^{-(1-v)x}\, \text{dx} = \left.\frac{e^{-(1-v)x}}{1-v}\right]_0^\infty$$

$$M_X(v) = \frac{1}{1-v}$$

$$we\ know\ that\ m_n = \left.\frac{d^n M_X(v)}{dv^n}\right|_{v=0}$$

$$m_1 = \left.\frac{dM_X(v)}{dv}\right|_{v=0}$$

$$= \left.\frac{1}{(1-v)^2}\right|_{v=0} = 1$$

8. Find the characteristic function of a uniformly distributed random variable X in the range[0,1] and hence find out m_1

Sol:

We know that the Probability density function for uniform random variable is

$$f_X(x) = \frac{1}{b-a} \quad a \le X \le b$$
$$= 0 \quad othewise$$

in the range[0,1] a=0, b=1

$$f_X(x) = 1 \quad 0 \le X \le 1$$
$$= 0 \quad othewise$$

The characteristic function is

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega X} f_X(x)\,\text{dx}$$

$$= \int_0^1 \frac{e^{j\omega X}}{j\omega}\,\text{dx}$$

$$= \frac{e^{j\omega X}}{j\omega}\Bigg]_0^1$$

$$\phi_X(\omega) = \frac{1}{j\omega}[e^{j\omega} - 1]$$

To find out m_1

$$m_n = (-j)^n \frac{d^n\phi_X(\omega)}{d\omega^n}\Bigg|_{\omega=0}$$

$$m_1 = (-j)\ \frac{d\phi_X(\omega)}{d\omega}\Bigg|_{\omega=0}$$

$$\frac{d\phi_X(\omega)}{d\omega} = \frac{d}{d\omega}\left|\frac{e^{j\omega}-1}{\omega}\right|$$

$$= \frac{1}{2}$$

9. A random variable X has a probability density

$$f_X(x) = \begin{cases} \frac{\pi}{16} cos\left(\frac{\pi x}{8}\right) \ , & -4 < x < 4 \\ 0 \ , & elsewhere \end{cases}$$

Find a)Mean value b)Second moment c)Variance

Sol: a) Mean value:

We know that $E[X] = \bar{X} = \int_{-\infty}^{\infty} x\, f_X\,(x)\,dx$

$$= \int_{-4}^{4} x \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16} \int_{-4}^{4} x \cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16}\left[\frac{x sin\frac{\pi}{8}x}{\frac{\pi}{8}}\Bigg|_{-4}^{4} - \int_{-4}^{4} \frac{\sin\frac{\pi}{8}x}{\frac{\pi}{8}} dx\right]$$

$$= \frac{\pi}{16}\left[\frac{8x}{\pi} sin\left(\frac{\pi x}{8}\right)\Bigg|_{-4}^{4} + \frac{64}{\pi^2} cos\left(\frac{\pi x}{8}\right)\Bigg|_{-4}^{4}\right]$$

$$= \frac{\pi}{16}\left[\frac{32}{\pi}\sin\frac{\pi}{2} + \frac{64}{\pi^2} cos\frac{\pi}{2} - \left[-\frac{32}{\pi} sin\left(-\frac{\pi}{2}\right) + \frac{64}{\pi^2} cos\left(-\frac{\pi}{2}\right)\right]\right]$$

$$= \frac{\pi}{16}\left[\frac{32}{\pi} - \frac{32}{\pi}\right]$$

$$= \frac{\pi}{16}(0)$$

$$\therefore\ Mean\ value = 0$$

b) Second Moment:

$$m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{-4}^{4} x^2 \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16} \int_{-4}^{4} x^2 \cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16}\left[\frac{x^2 sin\frac{\pi}{8}x}{\frac{\pi}{8}}\Bigg|_{-4}^{4} - \int_{-4}^{4} 2x \frac{\sin\frac{\pi}{8}x}{\frac{\pi}{8}} dx\right]$$

$$= \frac{\pi}{16}\left[\frac{8x^2}{\pi} sin\left(\frac{\pi x}{8}\right)\Big|_{-4}^{4} - \frac{16x}{\pi}\left(\frac{-cos\frac{\pi}{8}x}{\frac{\pi}{8}}\right)\Bigg|_{-4}^{4} + \int_{-4}^{4} 2\left(\frac{-cos\frac{\pi}{8}x}{\frac{\pi}{8}}\right) dx\right]$$

$$= \frac{\pi}{16}\left[\frac{256}{\pi} sin\frac{\pi}{2} + \frac{1024}{\pi^2} cos\frac{\pi}{2} - \frac{256}{\pi^2} sin\frac{\pi}{2}\right]$$

$$= \frac{\pi}{16}\left[\frac{256}{\pi} - \frac{256}{\pi^2}\right]$$

$$= 16 - \frac{16}{\pi}$$

$$= 16 - 5.0929$$

$$\therefore\ m_2 = 10.9071$$

c) Variance:

$$\sigma_x^2 = m_2 - {m_1}^2$$

$$= 10.9071 - 0$$

$$\therefore\ Variance = 10.9071$$

10. Calculate a)$E[X]$ b)$E[4X + 2]$ c)$E[X^2]$ of a random variable has a probability density

$$f_X(x) = \begin{cases} \frac{5}{4}(1 - x^4) \ , \ 0 < x \le 1 \\ 0 \ , \ elsewhere \end{cases}$$

Sol: a) E[X]:

$$E[X] = \int_{-\infty}^{\infty} x\, f_X\,(x)\, dx$$

$$= \int_{0}^{1} x\, \frac{5}{4}(1 - x^4)\, dx$$

$$= \frac{5}{4}\left[\int_0^1 x\,dx - \int_0^1 x^5\,dx\right]$$

$$= \frac{5}{4}\left[\frac{x^2}{2}\Big|_0^1 - \frac{x^6}{6}\Big|_0^1\right]$$

$$= \frac{5}{4}\left[\frac{1}{2} - \frac{1}{6}\right]$$

$$= \frac{5}{12}$$

$$\therefore\ E[X] = 0.4166$$

b) E$[4X + 2]$:

$$E[4X + 2] = 4E[X] + 2$$

$$= 4 \times (0.4166) + 2$$

$$= 1.6664 + 2$$

$$\therefore\ E[4X + 2] = 3.6664$$

c) E$[X^2]$:

We know that, $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$$= \int_0^1 x^2 \frac{5}{4}(1 - x^4)\,dx$$

$$= \frac{5}{4}\left[\int_0^1 x^2\,dx - \int_0^1 x^6\,dx\right]$$

$$= \frac{5}{4}\left[\frac{x^3}{3}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1\right]$$

$$= \frac{5}{4}\left[\frac{1}{3} - \frac{1}{7}\right]$$

$$= \frac{5}{4}\left[\frac{4}{21}\right]$$

$$= \frac{5}{21}$$

$$\therefore\ E[X^2] = 0.2380$$

11. Find the density function of the random variable X. If the characteristic function is given by

$$\phi_X(\omega) = \begin{cases} 1 - |\omega|\,, & |\omega| \le 1 \\ 0\,, & otherwise \end{cases}$$

Sol: Given,

$$\phi_X(\omega) = \begin{cases} 1 - |\omega|\,, & |\omega| \le 1 \\ 0\,, & otherwise \end{cases}$$

$$|\omega| \le 1 \Rightarrow (\omega \le 1)\ and\ (\omega \ge -1)$$

$$\Rightarrow -1 \le \omega \le 1$$

We know that,

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\phi_X(\omega)\, d\omega$$

$$\Rightarrow\ f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\,[1 - |\omega|]\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{-j\omega x}[1 - |\omega|]\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{0} e^{-j\omega x}\,(1 + \omega)\, d\omega + \int_{0}^{1} e^{-j\omega x}(1 - \omega)\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{0} e^{-j\omega x}\, d\omega + \int_{-1}^{0} \omega e^{-j\omega x}\, d\omega + \int_{0}^{1} e^{-j\omega x}\, d\omega - \int_{0}^{1} \omega e^{-j\omega x}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega x}}{-jx}\right]_{-1}^{0} + \frac{1}{2\pi}\left[\frac{\omega e^{-j\omega x}}{-jx} - \int_{-1}^{0}\frac{e^{-j\omega x}}{-jx}d\omega\right] + \frac{1}{2\pi}\left[\frac{e^{-j\omega x}}{-jx}\right]_{0}^{1}$$

$$- \frac{1}{2\pi}\left[\frac{\omega e^{-j\omega x}}{-jx} - \int_{0}^{1}\frac{e^{-j\omega x}}{-jx}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{-jx} - \frac{e^{jx}}{-jx}\right] + \frac{1}{2\pi}\left[\frac{\omega e^{-j\omega x}}{-jx} - \frac{e^{-j\omega x}}{(jx)^2}\right]_{-1}^{0} + \frac{1}{2\pi}\left[\frac{e^{-jx}}{-jx} - \frac{1}{-jx}\right]$$

$$- \frac{1}{2\pi}\left[\frac{\omega e^{-j\omega x}}{-jx} - \frac{e^{-j\omega x}}{(jx)^2}\right]_{0}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{-1 + e^{jx}}{jx}\right] + \frac{1}{2\pi}\left[0 - \frac{1}{(jx)^2} - \left(\frac{-e^{jx}}{-jx} - \frac{e^{jx}}{(jx)^2}\right)\right] + \frac{1}{2\pi}\left[\frac{1 - e^{jx}}{jx}\right]$$

$$- \frac{1}{2\pi}\left[\frac{e^{-jx}}{-jx} - \frac{e^{-jx}}{(jx)^2} - \left(0 - \frac{1}{(jx)^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1 + e^{jx} + 1 - e^{-jx}}{jx}\right] + \frac{1}{2\pi}\left[\frac{-e^{jx}}{jx} + \frac{e^{jx} - 1}{(jx)^2}\right] - \frac{1}{2\pi}\left[\frac{e^{-jx}}{-jx} + \frac{1 - e^{-jx}}{(jx)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{jx} - e^{-jx} - e^{jx} + e^{-jx}}{jx}\right] + \frac{1}{2\pi}\left[\frac{e^{jx} - 1 - 1 + e^{-jx}}{(jx)^2}\right]$$

$$= \frac{1}{2\pi}[0] + \frac{1}{2\pi}\left[\frac{e^{jx} + e^{-jx} - 2}{(jx)^2}\right]$$

$$= \frac{1}{2\pi} \times 2\left[\frac{\frac{e^{jx} + e^{-jx}}{2} - 1}{(jx)^2}\right]\left[\because \cos x = \frac{e^{jx} + e^{-jx}}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{cosx - 1}{(jx)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{cosx - 1}{-x^2}\right]$$

$$= \frac{1 - \cos x}{\pi x^2}$$

$$\therefore \ \boldsymbol{Density\ function};\ \boldsymbol{f_X(x)} = \frac{\mathbf{1} - \boldsymbol{\cos x}}{\boldsymbol{\pi x^2}}$$